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# Deformations of the boson $sp(4, R)$ representation and its subalgebras

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## Abstract

The boson representation of  $sp(4, R)$  algebra and two distinct deformations of it,  $sp_q(4, R)$  and  $sp_t(4, R)$ , are considered, as well as the compact and noncompact subalgebras of each. The initial as well as the deformed representations act in the same Fock space,  $\mathcal{H}$ , which is reducible into two irreducible representations acting in the subspaces  $\mathcal{H}_+$  and  $\mathcal{H}_-$  of  $\mathcal{H}$ . The deformed representation of  $sp_q(4, R)$  is based on the standard  $q$ -deformation of the boson creation and annihilation operators. The subalgebras of  $sp(4, R)$  (compact  $u(2)$  and noncompact  $u^\varepsilon(1, 1)$  with  $\varepsilon = 0, \pm$ ) are also deformed and their deformed representations are contained in  $sp_q(4, R)$ . They are reducible in the  $\mathcal{H}_+$  and  $\mathcal{H}_-$  spaces and decompose into irreducible representations. In this way a full description of the irreducible unitary representations of  $u_q(2)$  of the deformed ladder series  $u_q^0(1, 1)$  and of two deformed discrete series  $u_q^\pm(1, 1)$  are obtained. The other deformed representation,  $sp_t(4, R)$ , is realized by means of a transformation of the  $q$ -deformed bosons into  $q$ -tensors (spinor-like) with respect to the  $su_q(2)$  operators. All of its generators are deformed and have expressions in terms of tensor products of spinor-like operators. In this case, a deformed  $su_t(2)$  appears in a natural way as a subalgebra and can be interpreted as a deformation of the angular momentum algebra  $so(3)$ . Its representation in  $\mathcal{H}$  is reducible and decomposes into irreducible ones that yields a complete description of the same. The basis states in  $\mathcal{H}_+$ , which require two quantum labels, are expressed in terms of three of the generators of the  $sp(4, R)$  algebra and are labelled by three linked integer parameters.

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## 1. Introduction

Symplectic algebras enter in physical applications when operators that change the number of particles of the system are employed. One example is a description of collective vibrational excitations of a system of particles moving in an  $n$ -dimensional harmonic oscillator potential.

In this paper we start by considering the simplest two-dimensional case with  $sp(4, R)$  as its dynamical symmetry algebra.  $sp(4, R)$  is a noncompact group that is isomorphic to  $O(3, 2)$  [1]. A reduction from  $sp(4, R)$  to the  $u(2) = su(2) \oplus u(1) \sim so(3) \oplus o(2)$  subalgebra gives rise to a classification scheme with basis states that exhibit collective rotations. There are three possible reductions to representations of the noncompact  $u(1, 1) = su(1, 1) \oplus u(1)$  subalgebra that are important for a complete classification of the basis states of a system.

Although  $sp(4, R)$  is the simplest nontrivial case of a noncompact symplectic algebra, this structure is realized in various applications [2, 3]. It also serves as an example for exactly solvable test models [1]. The applications are related to different interpretations of the quantum numbers of the bosons used to create its representations. In addition to its use as the dynamical symmetry in some collective models of nuclear structure [4],  $sp(4, R)$  has been used for a complete classification of yrast-band energies in even–even nuclei [5]. And since it is rather easy to generalize  $sp(4, R)$  results to higher rank algebras [4], features uncovered for  $sp(4, R)$  have extended applications, the bosonization of other symplectic algebras being a case in point [6]. A further application of  $sp(4, R)$  is in the application of mapping methods [7], where the main purpose is to simplify the Hamiltonian of the initial problem [8]. In all such applications, a tensor realization of  $sp(4, R)$  algebra derived from the usual boson creation and annihilation operators, is most convenient.

In the last decade a lot of effort, from a purely mathematical as well as a physical point of view [9–11] has been concentrated on various deformations of the classical Lie algebras. The general feature of these deformations is that at some limit of the deformation parameter  $q$ , the  $q$ -algebra reverts back to a classical Lie algebra. More than one deformation can be realized for one and the same ‘classical’ algebra, which can be exploited in different physical applications. There are a lot of similarities between the classical Lie algebras and their deformations, especially with respect to the action spaces of their representations. The study of deformed algebras can also lead to a deeper understanding of the physical significance of the deformation.

In this paper we explore boson representations of  $sp(4, R)$  algebra. We begin with the well known representation of this algebra in terms of ‘classical’ boson creation and annihilation operators (section 1) and consider all the subalgebras and various ways to specify basis states by means of eigenvalues of the operators associated with them. We also introduce a deformation of this algebra in terms of standard  $q$ -bosons, and following the same procedure we investigate the enveloping algebra of  $sp(4, R)$  that is so obtained and explore the action of its generators on the basis, which remains the same (section 2). We obtain another deformation of the same algebra by transforming the  $q$ -deformed bosons into tensor operators with respect to the compact subgroup  $SU_q(2)$  defined in the previous section. In this case we use  $q$ -tensor products to obtain its generators, which are also tensor operators in respect to  $SU_q(2)$ . Their components form subalgebras in a natural way and the compact subalgebra  $su_t(2)$  that is so obtained can be interpreted as isomorphic to a deformation of the  $so(3)$  algebra (section 3). In the last section (section 4) we investigate a representation of the basis in terms of generators of  $sp_t(4, R)$  algebra, which introduces three quantum numbers for specifying states that map onto corresponding classical results.

## 2. Boson representations of $sp(4, R)$ algebra

Let us begin by recalling some features of the boson representation of  $sp(4, R)$  [4, 12]. The operators  $b_i^\dagger$  and  $b_i$ ,  $i = \pm 1$ , which satisfy Bose commutation relations

$$[b_i, b_k^\dagger] = \delta_{i,k} \quad [b_i^\dagger, b_k^\dagger] = [b_i, b_k] = 0 \quad (1)$$

are the natural language for a description of the two-dimensional harmonic oscillator [1]. They act in a Hilbert space  $\mathcal{H}$  with a vacuum  $|0\rangle$  so that  $b_i |0\rangle = 0$ . The scalar product in  $\mathcal{H}$  is chosen so that  $b_i^\dagger$  is the Hermitian conjugate of  $b_i$   $[(b_i^\dagger)^* = b_i]$  and  $\langle 0|0\rangle = 1$ . The vectors

$$|v_1, v_{-1}\rangle = \frac{(b_1^\dagger)^{v_1} (b_{-1}^\dagger)^{v_{-1}}}{\sqrt{v_1! v_{-1}!}} |0\rangle \tag{2}$$

where  $v_1, v_{-1}$  run over all non-negative integers form an orthonormal basis in  $\mathcal{H}$ . They are the common eigenvectors of the boson number operators  $N_1 = b_1^\dagger b_1$ ,  $N_{-1} = b_{-1}^\dagger b_{-1}$  and  $N = N_1 + N_{-1}$ :

$$\begin{aligned} N_1 |v_1, v_{-1}\rangle &= v_1 |v_1, v_{-1}\rangle \\ N_{-1} |v_1, v_{-1}\rangle &= v_{-1} |v_1, v_{-1}\rangle \\ N |v_1, v_{-1}\rangle &= v |v_1, v_{-1}\rangle \end{aligned} \tag{3}$$

where  $v = v_1 + v_{-1}$  and

$$N_i = N_i^* \quad [N_i, b_i^\dagger] = b_i^\dagger \quad [N_i, b_i] = -b_i \quad i = \pm 1. \tag{4}$$

The boson representation of  $sp(4, R)$  is given in a standard way by means of the operators  $F_{i,j} = b_i^\dagger b_j^\dagger$ ,  $G_{i,j} = F_{i,j}^* = b_i b_j$  and  $A_{i,j} = A_{j,i}^* = b_i^\dagger b_j + \frac{1}{2} \delta_{i,j}$  where  $i, j = \pm 1$  [12]. It is reducible and decomposes into two irreducible representations, each acting in the subspaces  $\mathcal{H}_+$  and  $\mathcal{H}_-$  of  $H$  labelled by the eigenvalue of the  $sp(4, R)$  invariant operator  $P = (-1)^N$ :

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \quad P|\varphi_\pm\rangle = \pm|\varphi_\pm\rangle \quad |\varphi_\pm\rangle \in \mathcal{H}_\pm.$$

In other words,  $\mathcal{H}_+$  is spanned by the vectors (2) with  $v = v_1 + v_{-1}$  even and  $\mathcal{H}_-$  with  $v$ -odd, respectively.

The maximal compact subgroup  $U(2)$  of  $Sp(4, R)$  can be generated by the Weyl generators  $A_{i,j}$  as well as by the well known equivalent system

$$\begin{aligned} I_+ &= b_1^\dagger b_{-1} & I_- &= I_+^* = b_{-1}^\dagger b_1 \\ I_0 &= I_0^* = \frac{1}{2}(b_1^\dagger b_1 - b_{-1}^\dagger b_{-1}) & N & \end{aligned} \tag{5}$$

that satisfies the commutation relations

$$[I_0, I_\pm] = \pm I_\pm \quad [I_+, I_-] = 2I_0 \quad [N, I_\pm] = 0 \quad [N, I_0] = 0. \tag{6}$$

The operators  $I_0, I_\pm$  close the algebra  $su(2) \sim so(3)$ . The operator  $N$  generates  $u(1)$  and plays the role of the first-order invariant (6) of  $U(2) = SU(2) \otimes U(1)$ . Each of the  $\mathcal{H}_+$  and  $\mathcal{H}_-$  subspaces decompose into a direct sum of eigensubspaces of  $N$ , defined by the condition that  $v$  is fixed:

$$\mathcal{H}_+ = \bigoplus_v \mathcal{H}_v^+ \quad \mathcal{H}_- = \bigoplus_v \mathcal{H}_v^-. \tag{7}$$

An irreducible unitary representation (IUR) of  $U(2)$  is realized in each  $\mathcal{H}_v^\pm$  space.

Another option for labelling the basis vectors (2) is the eigenvalues of the second-order Casimir operator of  $SU(2)$

$$I^2 = \frac{1}{2}(I_+ I_- + I_- I_+) + I_0 I_0 = \frac{N}{2} \left( \frac{N}{2} + 1 \right) \tag{8}$$

and its third projection,  $I_0$

$$I^2 |i, i_0\rangle = i(i+1) |i, i_0\rangle \quad I_0 |i, i_0\rangle = i_0 |i, i_0\rangle$$

where from (8)  $i = \frac{v}{2} = \frac{1}{2}(v_1 + v_{-1})$ ,  $i_0 = \frac{1}{2}(v_1 - v_{-1})$  and

$$|i, i_0\rangle = \frac{(b_1^\dagger)^{(i+i_0)} (b_{-1}^\dagger)^{(i-i_0)}}{\sqrt{(i+i_0)! (i-i_0)!}} |0\rangle \equiv |v_1, v_{-1}\rangle. \tag{9}$$

$\nu/i_0$	3	2	1	0	-1	-2	-3
0				$ 0\rangle$			
2			$ 2,0\rangle$	$ 1,1\rangle$	$ 0,2\rangle$		
4		$ 4,0\rangle$	$ 3,1\rangle$	$ 2,2\rangle$	$ 1,3\rangle$	$ 0,4\rangle$	
6	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 1.  $\mathcal{H}_+$  space.

$\nu/i_0$	$\frac{7}{2}$	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$	$-\frac{7}{2}$
1				$ 1,0\rangle$	$ 0,1\rangle$			
3			$ 3,0\rangle$	$ 2,1\rangle$	$ 1,2\rangle$	$ 0,3\rangle$		
5		$ 5,0\rangle$	$ 4,1\rangle$	$ 3,2\rangle$	$ 2,3\rangle$	$ 1,4\rangle$	$ 0,5\rangle$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 2.  $\mathcal{H}_-$  space.

Applying the raising and lowering operators  $I_{\pm}$

$$I_{\pm}|i, i_0\rangle = \sqrt{(i \mp i_0)(i \pm i_0 + 1)}|i, i_0\rangle \tag{10}$$

to the lowest  $|i, -i\rangle$  (highest  $|i, i\rangle$ ) weight state  $\nu$  times we obtain all the basis states of a given representation.

We are also interested in the noncompact content of the boson representation of  $sp(4, R)$ .

- (1) A reducible unitary representation ('ladder series' [13])  $u^0(1, 1)$  of the algebra  $u(1, 1)$  with Weyl generators  $b_1^\dagger b_1, b_1^\dagger b_{-1}^\dagger, -b_{-1} b_1, -b_{-1} b_{-1}^\dagger$  acts in  $\mathcal{H}$ . The first-order Casimir operator of  $U^0(1, 1)$  is essentially the operator  $I_0$

$$C_1^0 = b_1^\dagger b_1 - b_{-1} b_{-1}^\dagger = 2I_0 - 1$$

and hence the reduction of the ladder series into IURs (ladders) can be carried out using  $I_0$ . The spaces  $\mathcal{H}_{\pm}$  decompose into direct sums of eigenspaces of  $I_0$  labelled by  $i_0 = \frac{1}{2}(\nu_1 - \nu_{-1})$

$$\mathcal{H}_+ = \bigoplus_{i_0} \mathcal{H}_{i_0}^+ \quad \mathcal{H}_- = \bigoplus_{i_0} \mathcal{H}_{i_0}^- \tag{11}$$

An irreducible representation (a ladder) of the  $u^0(1, 1)$  is induced in each  $\mathcal{H}_{i_0}^{\pm}$  space. The operators  $N = N_1 + N_{-1}$  and  $I_0 = \frac{1}{2}(N_1 - N_{-1})$  can be considered as another complete set of operators, both diagonal in the basis (2) and therefore uniquely specifying the states. We can represent this fact with the pyramids given in figure 1, where the rows representing the IUR of  $U(2)$  are labelled by  $\nu$  and the columns representing the ladders of  $u^0(1, 1)$  by  $i_0$ . Each cell corresponds to one of the states  $|\nu_1, \nu_{-1}\rangle$  defined by (2). For the  $\mathcal{H}_+$  space we have figure 1 and for  $\mathcal{H}_-$  figure 2.

The set of operators  $F_0 \equiv F_{1,-1} = b_1^\dagger b_{-1}^\dagger, G_0 \equiv G_{1,-1} = b_1 b_{-1}$  and  $A_0 = \frac{1}{2}(N + 1)$  give a representation  $su^0(1, 1)$  of the  $su(1, 1)$  algebra. They commute in the following way:

$$[A_0, F_0] = F_0 \quad [A_0, G_0] = -G_0 \quad [F_0, G_0] = -2A_0.$$

By adding the operator  $I_0$  we obtain the  $u^0(1, 1) = su^0(1, 1) \oplus u^0(1)$  extension (5).

The second-order Casimir invariant of this subgroup is given by

$$C_2(SU^0(1, 1)) = (A_0)^2 - \frac{1}{2}(F_0 G_0 + G_0 F_0) = (I_0 + \frac{1}{2})(I_0 - \frac{1}{2}). \tag{12}$$

The quadratic equation  $(i_0 + \frac{1}{2})(i_0 - \frac{1}{2}) = \phi(\phi + 1)$  has two solutions for  $\phi$ :  $\phi_1 = i_0 - \frac{1}{2}$  and  $\phi_2 = -i_0 - \frac{1}{2}$ . Thus the discrete positive series  $D^+$  of IURs of  $su(1, 1)$  is realized

for the real negative values of  $\phi_{1,2}$  [12]. The corresponding spectra of  $\phi_1(i_0 \leq 0)$  and  $\phi_2(i_0 \geq 0)$  are

$\phi_1$	=	$-\frac{1}{2},$	$-1,$	$-\frac{3}{2},$	$\dots$
$i_0$	=	$0,$	$-\frac{1}{2},$	$-1,$	$\dots$
$\phi_2$	=	$-\frac{1}{2},$	$-1,$	$-\frac{3}{2},$	$\dots$
$i_0$	=	$0,$	$\frac{1}{2},$	$1,$	$\dots$

In the framework of  $su^0(1, 1)$  a degeneracy takes place—the same IUR of  $su(1, 1)$  is realized in  $\mathcal{H}_{i_0}^\pm$  and in  $\mathcal{H}_{-i_0}^\pm$ . This degeneracy is removed by the operator  $I_0$ , i.e., after the extension of  $su^0(1, 1)$  to  $u^0(1, 1)$ . In each representation of  $D^+$  the spectrum of  $A_0 = \frac{1}{2}(N + 1)$  is given by  $\alpha_0 = \frac{1}{2}(v + 1) = -\phi_i, -\phi_i + 1, -\phi_i + 2 \dots, i = 1, 2$ .

- (2) Next we consider two mutually complementary representations  $su^+(1, 1)$  and  $su^-(1, 1)$  of the algebra  $su(1, 1) \subset sp(4, R)$  acting in  $\mathcal{H}$ . They are given by the operators  $F_\pm = \frac{1}{2}F_{\pm 1, \pm 1}, G_\pm = \frac{1}{2}G_{\pm 1, \pm 1}$  and  $A_\pm = \frac{1}{2}(N_{\pm 1} + \frac{1}{2})$ , respectively, with commutation relations

$$[A_\pm, F_\pm] = F_\pm \quad [A_\pm, G_\pm] = -G_\pm \quad [F_\pm, G_\pm] = -2A_\pm.$$

It is simple to see that each of the generators of  $SU^+(1, 1)$  commutes with all the generators of the other  $SU^-(1, 1)$  subgroup. The second-order Casimir operators of the  $SU^\pm(1, 1)$  are

$$C_2(SU^\pm(1, 1)) = (A_0^\pm)^2 - \frac{1}{2}(F_\pm G_\pm + G_\pm F_\pm) = -\frac{3}{16}. \tag{13}$$

The equation  $\phi(\phi + 1) = -\frac{3}{16}$  has two solutions:  $\phi_1^\pm = -\frac{1}{4}$  and  $\phi_2^\pm = -\frac{3}{4}$ . Therefore, two unitary representations from the  $D^+$  series are realized. The corresponding spectra of the eigenvalues  $\alpha_\pm = \frac{1}{2}(v_{\pm 1} + \frac{1}{2})$  of the operators  $A_\pm$  for  $\phi_1^\pm = -\frac{1}{4}$  are given by

$\alpha_\pm$	=	$\frac{1}{4},$	$\frac{5}{4},$	$\frac{9}{4},$	$\dots$
$v_{\pm 1}$	=	$0,$	$2,$	$4,$	$\dots$

(14)

and for  $\phi_2^\pm = -\frac{3}{4}$  by

$\alpha_\pm$	=	$\frac{3}{4},$	$\frac{7}{4},$	$\frac{11}{4},$	$\dots$
$v_{\pm 1}$	=	$1,$	$3,$	$5,$	$\dots$

(15)

In this case the addition of the operators  $N_{\mp 1}$  considered as generators of the representations  $U^\mp(1)$  of the group  $U(1)$ , extend  $su^\pm(1, 1)$  to the  $u^\pm(1, 1) = su^\pm(1, 1) \oplus u^\mp(1)$ .  $N_{\mp 1}$  act as first-order Casimir operators of  $U^\pm(1, 1)$ . The spaces  $\mathcal{H}_+$  and  $\mathcal{H}_-$  are decomposed into direct sums of eigensubspaces of  $N_{-1}$  and  $N_1$  as follows:

$$\begin{aligned} \mathcal{H}_+ &= \left( \bigoplus_{k=0}^\infty \mathcal{H}_{v_{\mp 1}=2k}(\phi = -\frac{1}{4}) \right) \oplus \left( \bigoplus_{k=1}^\infty \mathcal{H}_{v_{\mp 1}=2k+1}(\phi = -\frac{3}{4}) \right) \\ \mathcal{H}_- &= \left( \bigoplus_{k=0}^\infty \mathcal{H}_{v_{\mp 1}=2k}(\phi = -\frac{3}{4}) \right) \oplus \left( \bigoplus_{k=1}^\infty \mathcal{H}_{v_{\mp 1}=2k+1}(\phi = -\frac{1}{4}) \right). \end{aligned} \tag{16}$$

In each  $\mathcal{H}_{v_{\mp 1}}^\pm(\phi_i), i = 1, 2$  a IUR of  $u(1, 1)$  is realized. The degeneracy which takes place on the level of  $su(1, 1)$  is completely removed after the extension to  $u^\pm(1, 1)$ . The subspaces  $\mathcal{H}_{v_{\mp 1}}^\pm(\phi_i), i = 1, 2$  are represented in figure 1 by the diagonals defined by the conditions  $v_{\mp 1}$  being fixed.

(3) Finally we can construct another representation of  $su(1, 1)$  by the simple sum of the generators of the  $SU^\pm(1, 1)$

$$\begin{aligned} F &= \frac{1}{2}(F_{1,1} + F_{-1,-1}) \\ G &= \frac{1}{2}(G_{1,1} + G_{-1,-1}) \\ A &= \frac{1}{2}(N_1 + N_{-1} + 1) \equiv A_0. \end{aligned}$$

### 3. $q$ -bosons and the quantum $sp_q(4, R)$ algebra

In this section using  $q$ -deformation of classical bosons, we are going to construct a  $q$ -deformation  $sp_q(4, R)$  of the boson representation of the  $sp(4, R)$  algebra [10], in the same manner as in the previous section. We start by deforming the operators  $b_i^\dagger$  and  $b_i$ ,  $i = \pm 1$ , by means of the transformation [14]:

$$a_i^\dagger = \sqrt{\frac{[N_i]}{N_i}} b_i^\dagger \quad a_i = \sqrt{\frac{[N_i + 1]}{N_i + 1}} b_i \quad (17)$$

where  $[X] \equiv \frac{q^X - q^{-X}}{q - q^{-1}}$ . Obviously  $(a_i^\dagger)^* = a_i$ . It is possible to interpret the deformation of the classical boson creation and annihilation operators  $b_i^\dagger$  and  $b_i$  where  $i = \pm 1$ , by analysing the expansion of the coefficients in (17) in terms of the deformation parameter  $\tau$ , introduced as  $q = e^\tau$ :

$$\frac{[N_i]}{N_i} = 1 + \frac{1}{6}(N_i^2 - 1)\tau^2 + \frac{1}{12}\left(\frac{1}{10}N_i^4 - \frac{1}{3}N_i^2 + \frac{7}{30}\right)\tau^4 + O(\tau^6). \quad (18)$$

In this case we have an infinite expansion containing all the even powers of the deformation parameter and also all the even powers of each of the classical operators  $N_i$  of the number of bosons.

From (17) it is easy to obtain the  $q$ -deformed commutation relations for the deformed oscillators

$$a_i a_i^\dagger - q^1 a_i^\dagger a_i = q^{-N_i} \quad (19)$$

$$a_i a_i^\dagger - q^{-1} a_i^\dagger a_i = q^{N_i} \quad (20)$$

$$[a_i, a_k^\dagger] = 0 \quad [a_i^\dagger, a_k^\dagger] = [a_i, a_k] = 0 \quad i \neq k.$$

In terms of the deformed boson operators, the basis vectors (2) are  $(a_i|0) = 0$

$$|v_1, v_{-1}\rangle = \frac{(a_1^\dagger)^{v_1} (a_{-1}^\dagger)^{v_{-1}}}{\sqrt{[v_1]! [v_{-1}]!}} |0\rangle \equiv \frac{(b_1^\dagger)^{v_1} (b_{-1}^\dagger)^{v_{-1}}}{\sqrt{v_1! v_{-1}!}} |0\rangle \quad (21)$$

where  $[X]! = [1][2][3] \cdots [X]$ . Obviously the spectra (3) of the operators  $N_i$ ,  $i = \pm 1$ , is preserved (3). It is easy to see that their relations with the  $q$ -deformed bosons are the same as (4)

$$[N_i, a_i] = -a_i \quad [N_i, a_i^\dagger] = a_i^\dagger. \quad (22)$$

The  $q$ -boson representation of a  $q$ -deformed algebra  $sp_q(4, R)$  acting in the Fock space  $\mathcal{H}$  can be realized by the operators:

$$\begin{aligned} F_{i,j}^q &= a_i^\dagger a_j^\dagger & G_{i,j}^q &= (F_{j,i}^q)^* = a_i a_j & i, j &= \pm 1 \\ J_\pm &= a_{\pm 1}^\dagger a_{\mp 1} & J_0 &= \frac{1}{2}(N_1 - N_{-1}) \equiv I_0 & N &= N_1 + N_{-1}. \end{aligned} \quad (23)$$

In this representation the raising and lowering operators  $F_{i,j}^q$ ,  $G_{i,j}^q$  and  $J_+$ ,  $J_- = J_+^*$  are deformed. The complete set of operators  $N$  and  $I_0$  used in the  $sp(4, R)$  case is retained after the deformation. This allows one to rewrite the basic states (21) in the form (compare with (9))

$$|v_1, v_{-1}\rangle = \frac{(a_1^\dagger)^{j+m} (a_{-1}^\dagger)^{j-m}}{\sqrt{[j+m]! [j-m]!}} |0\rangle$$

where

$$N|j, m\rangle = 2j|j, m\rangle \quad J_0|j, m\rangle = m|j, m\rangle \quad (j \equiv i, m \equiv i_0).$$

It can be checked directly that

$$\begin{aligned} [J_{\pm}, N] &= 0 & [J_0, N] &= 0 \\ [F_{i,j}^q, N] &= -2F_{i,j}^q & [G_{i,j}^q, N] &= 2G_{i,j}^q. \end{aligned}$$

It follows that in this case the operator  $P = (-1)^N$  also commutes with all the elements of the representation of  $sp_q(4, R)$  considered. In other words, the decomposition  $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$  remains the same after the deformation. Thus the  $q$ -boson  $sp_q(4, R)$  representation decomposes into two irreducible ones acting in  $\mathcal{H}_+$  and  $\mathcal{H}_-$ , respectively.

The reduction to subalgebras is the same as in the nondeformed case. In  $\mathcal{H}$  there is a reducible representation of  $u_q(2)$  given by  $N$  and the operators  $J_{\pm}, J_0$  which commute in the following way:

$$[J_0, J_{\pm}] = \pm J_{\pm} \quad [J_+, J_-] = [2J_0]. \quad (24)$$

Since the same operator  $N$  acts also as a first-order invariant of  $u_q(2)$ , the decomposition (7) of the spaces  $\mathcal{H}_+$  and  $\mathcal{H}_-$  remains after the deformation. In each of the  $\mathcal{H}_v^{\pm}$  spaces there also acts an IUR of  $su_q(2)$  generated by  $J_0, J_{\pm}$ . The second-order Casimir operator in this case is given by the operator

$$\begin{aligned} \mathcal{J}^2 &= \frac{1}{2}(J_+J_- + J_-J_+) + \frac{1}{2}([J_0][J_0 + 1] + [J_0][J_0 - 1]) \\ &= J_+J_- + [J_0][J_0 - 1] = J_-J_+ + [J_0][J_0 + 1] = \left[\frac{N}{2}\right] \left[\frac{N}{2} + 1\right]. \end{aligned} \quad (25)$$

Its eigenvalues equation has the form

$$\mathcal{J}^2|j, m\rangle = [j][j + 1]|j, m\rangle.$$

The action of the deformed  $J_{\pm}$  is

$$J_{\pm}|j, m\rangle = \sqrt{[j \pm m + 1][j \mp m]}|j, m\rangle. \quad (26)$$

By acting with  $J_{\mp}$  on the highest (lowest) weight states  $|j, j\rangle$  ( $|j, -j\rangle$ )  $2j$  times we obtain all the basis in the space  $\mathcal{H}_v^{\pm}$  of a given IUR of  $su_q(2)$  (a row in the diagrams in figure 1). So in the context of boson  $sp_q(4, R)$  representation, we have a full description of IURs of the deformed  $su_q(2)$  algebra.

Thus far we have focused on compact structures; we now turn to a consideration of noncompact cases.

(1) In the  $\mathcal{H}$  space, a deformation  $u_q^0(1, 1)$  [14] of the  $u^0(1, 1)$  algebra acts. It is generated by the operators

$$\begin{aligned} K_+^0 &= F_{1,-1}^q = a_1^{\dagger} a_{-1}^{\dagger} & K_-^0 &= G_{1,-1}^q = a_1 a_{-1} \\ K_0^0 &= \frac{1}{2}(N + 1) & J_0 & \end{aligned}$$

which have the following commutation rules:

$$\begin{aligned} [K_0^0, K_{\pm}^0] &= \pm K_{\pm}^0 & [K_+^0, K_-^0] &= -[2K_0^0] \\ [K_0^0, J_0] &= 0 & [K_{\pm}^0, J_0] &= 0. \end{aligned}$$

The reduction of  $\mathcal{H}$  to eigenspaces of the operator  $J_0 = I_0$  (11) is invariant with respect to the deformation. So the IURs (ladders) of  $u_q^0(1, 1)$  act in  $\mathcal{H}_m^{\pm}$ ,  $m \equiv i_0$ . IURs of



$su_q^0(1, 1) \subset u_q^0(1, 1)$  and generated by  $K_{\pm}^0, K_0^0$  also act in  $\mathcal{H}_m^{\pm}$ . The second-order Casimir invariant of  $SU_q^0(1, 1)$  is given by

$$\begin{aligned} (\mathbf{K}^0)^2 &= \frac{1}{2}([K_0^0][K_0^0 + 1] + [K_0^0][K_0^0 - 1]) - \frac{1}{2}(K_+^0 K_-^0 + K_-^0 K_+^0) \\ &\equiv [K_0^0][K_0^0 + 1] - K_-^0 K_+^0 \\ &\equiv [K_0^0][K_0^0 - 1] - K_+^0 K_-^0 = [J_0]^2 - \left[\frac{1}{2}\right]^2. \end{aligned} \tag{27}$$

The last expression is obtained with the help of the relations

$$\begin{aligned} K_+^0 K_-^0 &= \left[\frac{N}{2}\right]^2 - [J_0]^2 \\ [K_0^0][K_0^0 - 1] &= \left[\frac{N}{2}\right]^2 - \left[\frac{1}{2}\right]^2. \end{aligned}$$

Hence the eigenvalues of  $(\mathbf{K}^0)^2$  on the basis vectors are

$$(\mathbf{K}^0)^2 |j, m\rangle = ([m] + \left[\frac{1}{2}\right])([m] - \left[\frac{1}{2}\right]) |j, m\rangle.$$

The equation

$$([m] + \left[\frac{1}{2}\right])([m] - \left[\frac{1}{2}\right]) = \phi^q (\phi^q + 2 \left[\frac{1}{2}\right])$$

has two solutions for  $\phi^q$ :  $\phi_1^q = [m] - \left[\frac{1}{2}\right]$ ; and  $\phi_2^q = -[m] - \left[\frac{1}{2}\right]$ . Imposing  $\phi^q < 0$  we obtain for  $q > 0$  a description of the deformed discrete series  $D_q^+$  of  $su_q^0(1, 1)$ . The spectra simultaneously run by  $\phi_i^q$  and  $m$  are, respectively, for  $\phi_1^q < 0$  ( $m \leq 0$ )

$\phi_1^q =$	$-\left[\frac{1}{2}\right]$	$-2\left[\frac{1}{2}\right]$	$-[1] - \left[\frac{1}{2}\right]$	$-\left[\frac{3}{2}\right] - \left[\frac{1}{2}\right] \dots$
$m =$	$0$	$-\frac{1}{2}$	$-1$	$-\frac{3}{2} \dots$

and for  $\phi_2^q < 0$  ( $m \geq 0$ )

$\phi_1^q =$	$-\left[\frac{1}{2}\right]$	$-2\left[\frac{1}{2}\right]$	$-[1] - \left[\frac{1}{2}\right]$	$-\left[\frac{3}{2}\right] - \left[\frac{1}{2}\right] \dots$
$m =$	$0$	$\frac{1}{2}$	$1$	$\frac{3}{2} \dots$

A degeneracy of the same type as in the nondeformed case takes place because of the quadratic dependence on  $[J_0]$  in (27). We should, however, emphasize that in this case the degeneracy is also removed by the operator  $J_0$ , which remains nondeformed and acts as a first invariant of  $u_q^0(1, 1)$ . The spectrum of  $K_0^0 = \frac{1}{2}(N + 1)$ , which is also a nondeformed operator, is related to the nondeformed  $\phi$  ( $\phi^q \rightarrow_{q \rightarrow 1} \phi$ ), and so we have

$$\frac{1}{2}(v + 1) = -\phi, -\phi + 1, -\phi + 2, \dots$$

(2) At the end we will discuss the deformations  $u_q^{\pm}(1, 1)$  [14] of the two mutually complementary representations  $u^{\pm}(1, 1)$ , each realized by only one kind of  $q$ -boson. The operators

$$\begin{aligned} K_{\pm}^{\pm} &= \frac{1}{[2]} F_{\pm 1, \pm 1}^q = \frac{1}{[2]} a_{\pm 1}^{\dagger} a_{\pm 1}^{\dagger} & K_{\pm}^{\pm} &= \frac{1}{[2]} G_{\pm 1, \pm 1}^q = \frac{1}{[2]} a_{\pm 1} a_{\pm 1} \\ K_0^{\pm} &= \frac{1}{2}(N_{\pm 1} + \frac{1}{2}) & N_{\mp 1} & \end{aligned} \tag{28}$$

commute among themselves in the following way:

$$\begin{aligned} [K_0^{\pm}, K_{\pm}^{\pm}] &= \pm K_{\pm}^{\pm} & [K_{\pm}^{\pm}, K_{\mp}^{\pm}] &= -[2K_0^{\pm}]_2 \\ [K_0^{\pm}, N_{\mp 1}] &= 0 & [K_{\pm}^{\pm}, N_{\mp 1}] &= 0 \end{aligned}$$

where the notation  $[X]_m \equiv \frac{q^{mX} - q^{-mX}}{q^m - q^{-m}}$  applies. The nondeformed operators  $N_{\mp 1}$  extend the  $su_q^\pm(1, 1)$  to  $u_q^\pm(1, 1)$  and act as first-order Casimir invariants. The second-order Casimir invariants have a slightly modified form (compare with (27))

$$\begin{aligned} C_2\{SU_q^\pm(1, 1)\} &= [K_0^\pm]_2[K_0^\pm + 1]_2 - K_-^\pm K_+^\pm \\ &= [K_0^\pm]_2[K_0^\pm - 1]_2 - K_+^\pm K_-^\pm. \end{aligned}$$

In this case we have the following expressions:

$$\begin{aligned} K_-^\pm K_+^\pm &= \frac{1}{[2]^2} [N_\pm + 1][N_\pm + 2] \\ K_+^\pm K_-^\pm &= \frac{1}{[2]^2} [N_\pm][N_\pm - 1] \\ &= \frac{1}{[2]^2} \left\{ \left[ \frac{1}{2}(2N_\pm - 1) \right]^2 - \left[ \frac{1}{2} \right]^2 \right\} \\ [K_0^\pm]_2[K_0^\pm - 1]_2 &= \frac{1}{[2]^2} \left\{ \left[ \frac{1}{2}(2N_\pm - 1) \right]^2 - 1 \right\}. \end{aligned}$$

As a result we obtain

$$\begin{aligned} C_2\{SU_q^\pm(1, 1)\} &= (\mathbf{K}^\pm)^2 = \frac{1}{[2]^2} \left( \left[ \frac{1}{2} \right]^2 - 1 \right) \\ &= \left( \left[ \frac{1}{4} \right]_2 - \left[ \frac{1}{2} \right]_2 \right) \left( \left[ \frac{1}{4} \right]_2 + \left[ \frac{1}{2} \right]_2 \right). \end{aligned}$$

In this case the  $q$ -deformed equation

$$(\mathbf{K}^\pm)^2 = {}^q\varphi^\pm ({}^q\varphi^\pm + 2 \left[ \frac{1}{2} \right]_2)$$

has the solutions

$${}^q\varphi_1^\pm = \left[ \frac{1}{4} \right]_2 - \left[ \frac{1}{2} \right]_2 \quad {}^q\varphi_2^\pm = -\left[ \frac{1}{4} \right]_2 - \left[ \frac{1}{2} \right]_2$$

which means that in  $\mathcal{H}$  space a discrete series of two  $q$ -deformed representations of  $su_q(1, 1)$  is realized for each kind of  $q$ -boson (with index  $+1$  or  $-1$ ). The spectra of  $K_0^\pm$  correspond to the limit  ${}^q\varphi_i^\pm \xrightarrow{q \rightarrow 1} \varphi_i^\pm$ ,  $i = 1, 2$  and coincide with the spectra of the operators  $A_\pm$  in the nondeformed picture (see (14) and (15)). Here, again the extension from  $su_q^\pm(1, 1)$  to  $u_q^\pm(1, 1) = su_q^\pm(1, 1) \oplus u^\mp(1)$  is realized by adding the operators  $N_{\mp 1}$ . Thus the degeneracy is eliminated in the same way as in the nondeformed case. Now the spaces in which the irreducible boson representations of the classical  $u^\pm(1, 1)$  and  $q$ -deformed  $u_q^\pm(1, 1)$  act are the same and the decompositions (16) are the same, since we have

$$\begin{aligned} \mathcal{H}_{v_\mp} ({}^q\varphi_1^\pm = -\left[ \frac{1}{4} \right]_2 - \left[ \frac{1}{2} \right]_2) &\equiv \mathcal{H}_{v_\mp} (\alpha_1^\pm = -\frac{3}{4}) \\ \mathcal{H}_{v_\mp} ({}^q\varphi_2^\pm = \left[ \frac{1}{4} \right]_2 - \left[ \frac{1}{2} \right]_2) &\equiv \mathcal{H}_{v_\mp} (\alpha_2^\pm = -\frac{1}{4}). \end{aligned}$$

#### 4. Deformation in terms of $su_q(2)$ tensor operators

The  $q$ -deformed bosons  $a_i^\dagger$  and  $a_i$ ,  $i = \pm 1$ , are not components of tensor operators with respect to the standard  $su_q(2)$  defined in the previous section [9, 15]. However, the following nontrivial modification of the creation and annihilation operators for  $k = \pm 1$  introduced in equations (17) and (21):

$$t_k^\dagger = q^{\frac{k}{4}} a_k^\dagger q^{\frac{kN-k}{2}} \quad (29)$$

$$t_k = q^{-\frac{k}{4}} a_{-k} q^{-\frac{kN_k}{2}} \quad (30)$$

transforms these operators [17] into two-dimensional conjugated, spinor-like (of rank  $\frac{1}{2}$ ) tensors,  $(t_k^\dagger)^* = t_{-k}$ , with respect to  $su_q(2)$ . These deformations can be related to the classical bosons  $b_i^\dagger, b_i, i = \pm 1$ , by means of their transformations to  $q$ -deformed oscillators (17).

From the oscillator commutation relations (19) and (21) we obtain the following commutation relations:

$$\begin{aligned} [t_k, t_l^\dagger]_{q^\rho} &= q^{-\frac{k}{2}} \delta_{k,-l} q^{-2J_0} \\ [t_k^\dagger, t_l^\dagger]_{q^\sigma} &= [t_k, t_l]_{q^\sigma} = 0 \\ \rho &= \frac{l-k}{2} = -\sigma \quad l, k = \pm 1. \end{aligned} \tag{31}$$

In this case another  $q$ -deformation,  $sp_t(4, R)$ , of  $sp(4, R)$  algebra, with generalized Gauss decomposition

$$\mathcal{G} = g_- \oplus h \oplus g_+$$

is constructed by the tensor products of the fundamental oscillator representation for  $su_q(2)$  (29) and (30)

$$(t \otimes t)_m^l := T_m^l \supseteq g_+ \quad l = 1, m = 0, \pm 1 \tag{32}$$

$$(t^\dagger \otimes t^\dagger)_m^l := \tilde{T}_m^l \supseteq g_- \quad l = 1, m = 0, \pm 1 \tag{33}$$

$$(t^\dagger \otimes t)_m^l := L_m^l \supseteq h \quad l = 0, 1, m = -l, -l+1, \dots, l. \tag{34}$$

This  $sp_t(4, R)$  algebra [17] is decomposed in a natural way into a deformed compact subalgebra  $h = su_t(2) \otimes u_t(1)$  that is generated by the spherical tensors  $L_m^1 (m = 0, \pm 1)$  and  $L_0^0$  (34) and  $g_+$  and  $g_-$ , which are two  $q$ -nilpotent subalgebras containing the components of the two conjugated first-rank tensors  $T_m^1 (m = 0, \pm 1)$  (32) and  $\tilde{T}_m^1 (m = 0, \pm 1)$  (33). In this case the  $su_t(2)$  generated by the components of a first-rank tensor  $L_m^1 (m = 0, \pm 1)$  (34) can be interpreted [19] as isomorphic by construction to a deformation of  $so(3)$ —the classical algebra of the angular momentum. Using the  $q$ -deformed realization [15] of the Clebsh–Gordon coefficients for  $su_q(2)$  (23), we obtain the following explicit expressions for the operators (32), (33) and (34) in terms of the  $q$ -spinors (29) and (30):

$$\begin{aligned} T_1^1 &= t_1^\dagger t_1^\dagger = (\tilde{T}_{-1}^1)^* \\ T_{-1}^1 &= t_{-1}^\dagger t_{-1}^\dagger = (\tilde{T}_1^1)^* \\ T_0^1 &= q^{-\frac{1}{2}} \sqrt{[2]} t_1^\dagger t_{-1}^\dagger = (\tilde{T}_0^1)^* \end{aligned} \tag{35}$$

$$\begin{aligned} L_1^1 &= t_1^\dagger t_1 = q^{-\frac{1}{2}} J_+ q^{-J_0} = (L_{-1}^1)^* \\ L_{-1}^1 &= t_{-1}^\dagger t_{-1} = q^{\frac{1}{2}} J_- q^{-J_0} = (L_1^1)^*. \end{aligned} \tag{36}$$

The above eight operators are the  $q$ -tensor analogues of the deformed raising and lowering generators of  $sp_q(4, R)$ . Furthermore we consider the operators

$$\begin{aligned} \mathcal{N}_1 &= t_1^\dagger t_{-1} = q^{\frac{1}{2}} [N_1] q^{N_{-1}} & \mathcal{N}_{-1} &= t_{-1}^\dagger t_1 = q^{-\frac{1}{2}} [N_{-1}] q^{-N_1} \\ t_{-1} t_1^\dagger &= q^{\frac{1}{2}} [N_1 + 1] q^{N_{-1}} & t_1 t_{-1}^\dagger &= q^{-\frac{1}{2}} [N_{-1} + 1] q^{-N_1} \end{aligned} \tag{37}$$

obtained by means of the substitutions (29) and (30). It must be noted that in this case these are deformed operators and do not have expression in terms of classical bosons unlike the boson number operators,  $N_1$  and  $N_{-1}$ , used in the case of  $sp_q(4, R)$ .

By means of an expansion similar to the one introduced in (18) it is easy to verify the mixing of two kinds of oscillators  $k = \pm 1$  introduced through the use of tensor operators

$$\begin{aligned} [N_k] q^{\pm N_{-k}} &= N_k \pm N_k N_{-k} \tau + \frac{1}{6} N_k (3N_{-k}^2 + N_k^2 - 1) \tau^2 \\ &\pm \frac{1}{6} N_k N_{-k} (N_k^2 - 1 + N_{-k}^2) \tau^3 + O(\tau^4). \end{aligned} \tag{38}$$

It is simple to see that the operators  $\mathcal{N}_1$  and  $\mathcal{N}_{-1}$  belong to the enveloping algebra of the classical oscillators. In (38) all the powers of the deformation parameter  $\tau$  and all the degrees of the two ‘classical’ operators  $N_k = b_k^\dagger b_k$  for  $k = \pm 1$  appear. Using (17), (29), and (30) we have the following relations:

$$\begin{aligned} [N_k, t_k^\dagger] &= t_k^\dagger & [N_k, t_{-k}] &= -t_{-k} \\ [N_k, t_{-k}^\dagger] &= 0 & [N_k, t_k] &= 0 \end{aligned}$$

and as a result we have the correct tensor properties

$$[J_0, t_k^\dagger] = \frac{k}{2} t_k^\dagger \quad [J_0, t_k] = \frac{k}{2} t_k.$$

The third component  $L_0^1$  and scalar operator  $L_0^0$  are obtained as

$$\begin{aligned} L_0^1 &= \frac{1}{[2]}(q[N_1]q^{N_1} - q^{-1}[N_{-1}]q^{-N_1}) = \frac{1}{[2]}(q[N_1][N_{-1} + 1] - q^{-1}[N_{-1}][N_1 + 1]) \\ L_0^0 &= ([N_1]q^{N_1} + [N_{-1}]q^{-N_1}) = [N]. \end{aligned} \tag{39}$$

Using (31) we find the commutation relations (32)–(34)

$$[L_{\pm 1}^1, L_{\mp 1}^1] = [2]L_0^1 q^{-2J_0}; [L_0^1, L_{\pm 1}^1] = \pm q^{\mp 1} L_{\pm 1}^1 q^{-2J_0}. \tag{40}$$

From (40) it is obvious that the components of the first-rank tensors  $L_m^1$  ( $m = 0, \pm 1$ ) close in a natural way on another deformation,  $su_t(2)$ , of the classical  $su(2)$ . The scalar operator  $L_0^0 = [N]$  commutes with all the components of the first-rank tensors  $L_m^1$  ( $m = 0, \pm 1$ )

$$[[N], L_m^1] = 0$$

and yields decomposition  $u_t(2) = su_t(2) \oplus u_q(1)$  with a first-order Casimir invariant  $[N]$ . The second-order Casimir operator for  $SU_t(2)$  is calculated as the scalar product

$$-\sqrt{[3]}(\mathbf{L} \otimes \mathbf{L})^0 = qL_{-1}L_{+1} + q^{-1}L_{+1}L_{-1} - L_0L_0 = \frac{1}{[2]}[N][N + 2]. \tag{41}$$

For completeness we present all the other commutation relations of the tensor operators (32)–(34) in a slightly different form than in [16]. The commutators of  $L_{\pm 1}^1$  with the pair raising and lowering operators define their transformation properties in respect to the  $q$ -deformed  $so(3)$  subalgebra

$$\begin{aligned} [L_{\pm 1}^1, \tilde{T}_m^1] &= \pm q^{-(m \pm 1)} \sqrt{[1 \mp m][1 \pm m + 1]} \tilde{T}_{m \pm 1}^1 q^{-2J_0} \\ [L_{\pm 1}^1, T_m^1] &= \pm q^{-(m \pm 2)} \sqrt{[1 \mp m][1 \pm m + 1]} T_{m \pm 1}^1 q^{-2J_0}. \end{aligned} \tag{42}$$

Hence the operators  $\tilde{T}_m^1$  and  $T_m^1$ ,  $m = 0, \pm 1$  form two conjugated vectors with respect to  $so(3)$  subalgebra. From the following commutators for  $k = \pm 1$  and  $m = \pm 1$ :

$$\begin{aligned} [\tilde{T}_m^1, \mathcal{N}_k]_{q^{2k}} &= \delta_{m,-k} q^{\frac{3}{2}k} [2] \tilde{T}_m^1 q^{-2J_0} \\ [\mathcal{N}_k, T_m^1]_{q^{2k}} &= \delta_{m,k} q^{-\frac{k}{2}} [2] T_m^1 q^{-2J_0} \\ [\tilde{T}_0^1, \mathcal{N}_k]_{q^{2k}} &= q^{\frac{k}{2}} \tilde{T}_0^1 q^{-2J_0} & [\mathcal{N}_k, T_0^1]_{q^{2k}} &= q^{\frac{k}{2}} T_0^1 q^{-2J_0} \end{aligned} \tag{43}$$

it is easy to obtain the commutation relations of  $\tilde{T}_m^1$  and  $T_m^1$  ( $m = 0, \pm 1$ ) with the operators (39).

The pair operators  $T_m^1, \tilde{T}_m^1$  generate the two  $q$ -nilpotent algebras  $g_+, g_-$  and fulfil the following  $q$ -commutation relations:

$$[T_{m_1}^1, T_{m_2}^1]_{q^{2(m_1 - m_2)}} = 0 \quad [\tilde{T}_{m_1}^1, \tilde{T}_{m_2}^1]_{q^{2(m_1 - m_2)}} = 0. \tag{44}$$

The commutation relations between the  $T_m^1$  and  $\tilde{T}_m^1$  close in terms of the components of the angular momentum  $q$ -analogue (33). The subset with  $m_1 + m_2 \neq 0$  can be presented in a unified way as

$$[T_{m_1}^1, \tilde{T}_{m_2}^1]_{q^{2(m_2 - m_1)}} = -q^{-m_1} \sqrt{[2][2(m_2 - m_1)]} L_{m_1 + m_2}^1 q^{-2J_0} \tag{45}$$

while for  $m_1 + m_2 = 0$  we obtain

$$\begin{aligned} [T_1^1, \tilde{T}_1^1]_{q^{-4}} &= -[2]\{q^{-2}q^{-2J_0} + q^{\frac{1}{2}}[2]\mathcal{N}_1\}q^{-2J_0} \\ [T_0^1, \tilde{T}_0^1] &= [2][N+1]q^{-2J_0} \\ [T_{-1}^1, \tilde{T}_{-1}^1]_{q^4} &= -[2]\{q^2q^{-2J_0} + q^{-\frac{1}{2}}[2]\mathcal{N}_{-1}\}q^{-2J_0}. \end{aligned} \quad (46)$$

In the limit  $q \rightarrow 1$  these reproduce the commutation relations of ‘classical’  $sp(4, R)$  algebra, which has a lot of interesting applications in nuclear physics. It should be noted that in this case we do not obtain a simple generalization of the noncompact  $su^\varepsilon(1, 1)$  ( $\varepsilon = 0, \pm$ ) subalgebras of  $sp(4, R)$  as is the case with  $q$ -bosons. By analysing relations (43) and (46) it can be seen that they close in the enveloping algebras of the respective classical  $u^\varepsilon(1, 1)$  ( $\varepsilon = 0, \pm$ ). Actually this is a general property of this tensor deformation of  $sp(4, R)$  because of the appearance of the  $q^{-2J_0} \xrightarrow{q \rightarrow 1} 1$  factors on the right-hand side of all commutation relations.

The problem of eliminating this is solved in [17].

Working in terms of tensor operators makes the evaluation of the most general  $sp_t(4, R)$  invariant operator with respect to the  $q$ -deformed  $so(3)$  subalgebra quite simple. It is constructed as a linear combination of the scalar products of (32)–(34) that preserve  $J_0$

$$\begin{aligned} S_2 &= s_1 T_k^1 \cdot \tilde{T}_{-k}^1 + s_2 \tilde{T}_k^1 \cdot T_{-k}^1 + s_3 L_k^1 \cdot L_{-k}^1 + s_4 [N]^2 \\ &= s_1 [N][N-1] + s_2 [N+2][N+3] + s_3 [N][N+2] + s_4 [N]^2. \end{aligned} \quad (47)$$

From this expression it is clear that four additional phenomenological parameters ( $s_i$  with  $i = 1, 2, 3, 4$ ) together with the deformation parameter are introduced in the invariant. This allows for a larger variety of interactions in the corresponding Hamiltonian problem.

## 5. The basis states in the case of $q$ -tensor $sp_t(4, R)$ algebra

Now consider  $\mathcal{H}$  as the space of the action of the  $q$ -deformed tensor representation of  $sp_t(4, R)$  described in the previous section. In terms of the spinor-like creation and annihilation operators, (29) and (30), the basic states (2) have the form

$$|v_1, v_{-1}\rangle = q^{-\frac{1}{4}(v_1 - v_{-1}) - \frac{1}{2}v_1 v_{-1}} \frac{(t_1^\dagger)^{v_1} (t_{-1}^\dagger)^{v_{-1}}}{\sqrt{[v_1]![v_{-1}]!}} |0\rangle \quad (48)$$

which introduces a dependence on  $q$ . It is easy to check that the operator  $P = (-1)^N$  commutes with all the generators, (32)–(34), so in the  $q$ -deformed tensor case the  $sp_t(4, R)$  representation is also reducible and splits into two irreducible ones acting in the  $\mathcal{H}_+$  and  $\mathcal{H}_-$  subspaces.

In what follows we will consider the space  $\mathcal{H}_+$  to have  $v = v_1 + v_{-1}$  even. The states of  $\mathcal{H}_-$  can be obtained from the ones in  $\mathcal{H}_+$  with the help of the operators  $t_k^\dagger$  and  $t_k$  ( $k = \pm 1$ ). The later can be considered to be the odd generators of the superalgebraic extension of even  $sp_t(4, R)$  [17].

Looking forward to future applications we represent the basic states  $|v_1, v_{-1}\rangle \in \mathcal{H}_+$  as

$$|n_1, n_0, n_{-1}\rangle = \eta(n_1, n_0, n_{-1}) (T_1^1)^{n_1} (T_0^1)^{n_0} (T_{-1}^1)^{n_{-1}} |0\rangle \equiv |v_1, v_{-1}\rangle \quad (49)$$

where  $n_i$  ( $i = 0, \pm 1$ ) are integers restricted by the linkages

$$v_1 = 2n_1 + n_0 \quad v_{-1} = 2n_{-1} + n_0 \quad (50)$$

and  $\eta(n_1, n_0, n_{-1})$  is the normalization factor given by

$$\eta(n_1, n_0, n_{-1}) = \frac{q^{-\frac{1}{2}(n_1 - n_{-1}) - n_1(n_0 + n_{-1}) - n_{-1}(n_0 + n_1)}}{\sqrt{[2]^{n_0} [2n_1 + n_0]! [2n_{-1} + n_0]!}}. \quad (51)$$

This representation of the basic states is useful for a consideration of the appropriate mapping procedures [18] for  $sp_t(4, R)$  algebra. It should be noted that in this case we use the

$\nu/i_0$	3	2	1	0	-1	-2	-3
0				$ 0, 0, 0\rangle$			
2			$ 1, 0, 0\rangle$	$ 0, 1, 0\rangle$	$ 0, 0, 1\rangle$		
4		$ 2, 0, 0\rangle$	$ 1, 1, 0\rangle$	$ 1, 0, 1\rangle$	$ 0, 1, 1\rangle$	$ 0, 0, 2\rangle$	
6	$ 3, 0, 0\rangle$	$ 2, 1, 0\rangle$	$ 2, 0, 1\rangle$	$ 1, 1, 1\rangle$	$ 1, 0, 2\rangle$	$ 0, 1, 2\rangle$	$ 0, 0, 3\rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 3.

$\nu/i_0$	3	2	1	0	-1	-2	-3
0				$ 0, 0, 0\rangle$			
2			$ 1, 0, 0\rangle$	$ 0, 1, 0\rangle$	$ 0, 0, 1\rangle$		
4		$ 2, 0, 0\rangle$	$ 1, 1, 0\rangle$	$ 0, 2, 0\rangle$	$ 0, 1, 1\rangle$	$ 0, 0, 2\rangle$	
6	$ 3, 0, 0\rangle$	$ 2, 1, 0\rangle$	$ 1, 2, 0\rangle$	$ 0, 3, 0\rangle$	$ 0, 2, 1\rangle$	$ 0, 1, 2\rangle$	$ 0, 0, 3\rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 4.

ordering (49) of the components of the vector  $T_m^1$  ( $m = 0, \pm 1$ ) (32). Other orderings can be obtained from this by means of the commutation relations (44). The  $q$ -factors that will appear in such a result can be incorporated into the normalization coefficients. Keeping in mind (3) and (37) it is obvious that

$$\mathcal{N}_1|v_1, v_{-1}\rangle = [v_1]q^{v_1}|v_1, v_{-1}\rangle \quad \mathcal{N}_{-1}|v_1, v_{-1}\rangle = [v_{-1}]q^{-v_1}|v_1, v_{-1}\rangle \tag{52}$$

$$[N]|v_1, v_{-1}\rangle = [v]|v_1, v_{-1}\rangle. \tag{53}$$

Therefore, passing to representation (49) and in view of (50) one finds

$$\begin{aligned} \mathcal{N}_1|n_1, n_0, n_{-1}\rangle &= [2n_1 + n_0]q^{2n_1+n_0}|n_1, n_0, n_{-1}\rangle \\ \mathcal{N}_{-1}|n_1, n_0, n_{-1}\rangle &= [2n_{-1} + n_0]q^{-2n_1-n_0}|n_1, n_0, n_{-1}\rangle \end{aligned}$$

and

$$[N]|n_1, n_0, n_{-1}\rangle = [2n]|n_1, n_0, n_{-1}\rangle.$$

Note that

$$2n = 2n_1 + 2n_0 + 2n_{-1} = v \quad j_0 = \frac{1}{2}(v_1 - v_{-1}) = n_1 - n_{-1}.$$

Now we can consider two extreme cases for the possible values of the additional quantum number  $n_0$ .

- (1)  $n_0$  takes on minimal values. Since we are in  $\mathcal{H}_+$  space the integers  $v_1$  and  $v_{-1}$  are simultaneously even or odd. If  $v_1$  and  $v_{-1}$  are even ( $\min n_0 = 0$ ) with the help of (31) we obtain from (48)

$$\begin{aligned} |n_1, 0, n_{-1}\rangle &= q^{-\frac{1}{2}(n_1-n_{-1})-2n_1n_{-1}} \frac{(T_1^1)^{n_1} (T_{-1}^1)^{n_{-1}}}{\sqrt{[2n_1]![2n_{-1}]!}} |0\rangle \\ &= q^{-\frac{1}{4}(v_1-v_{-1})-\frac{1}{2}v_1v_{-1}} \frac{(T_1^1)^{\frac{v_1}{2}} (T_{-1}^1)^{\frac{v_{-1}}{2}}}{\sqrt{[v_1]![v_{-1}]!}} |0\rangle. \end{aligned} \tag{54}$$

If  $v_1$  and  $v_{-1}$  are odd ( $\min n_0 = 1$ ) we find that

$$\begin{aligned} |n_1, 1, n_{-1}\rangle &= q^{-\frac{1}{2}(3n_1+n_{-1})-2n_1n_{-1}} \frac{(T_1^1)^{n_1} T_0^1 (T_{-1}^1)^{n_{-1}}}{\sqrt{[2][2n_1+1]![2n_{-1}+1]!}} |0\rangle \\ &= q^{-\frac{1}{4}(v_1-v_{-1}-2)-\frac{1}{2}v_1v_{-1}} \frac{(T_1^1)^{\frac{v_1-1}{2}} T_0^1 (T_{-1}^1)^{\frac{v_{-1}-1}{2}}}{\sqrt{[2][v_1]![v_{-1}]!}} |0\rangle. \end{aligned}$$

In this way the  $q$ -deformed spinors are coupled to maximal degrees in  $n_1$  and  $n_{-1}$  for the components  $T_1^1$  and  $T_{-1}^1$ , respectively. Representing the basis states in  $\mathcal{H}_+$  as  $|n_1, n_0, n_{-1}\rangle$  vectors, in the case of  $\min n_0 = 0$  or  $1$  we can redraw the pyramid in figure 1 as presented in figure 3.

From figure 3 it is easy to see that we can obtain each state from the left (right) diagonals of the pyramid by the action on the minimal-weight state of the raising operators  $T_1(T_{-1})$ , respectively.

- (2)  $n_0$  takes on maximal values. In this case we have  $n_{-1} = 0$  or  $n_1 = 0$  at  $\nu_1 \neq \nu_{-1}$  and  $n_{-1} = n_1 = 0$  at  $\nu_1 = \nu_{-1}$ . There are two possibilities as presented in figure 4.

On the left-hand side of figure 1, where  $\nu_1 \geq \nu_{-1}$  and the coupling of  $T_0^1(n_{-1} = 0, \max n_0 = \nu_{-1})$  is to the maximal degree

$$\begin{aligned} |n_1, n_0, 0\rangle &= q^{-\frac{1}{2}n_1 - n_1 n_0} \frac{(T_1^1)^{n_1} (T_0^1)^{n_0}}{\sqrt{[2]^{n_0} [2n_1 + n_0]! [n_0]!}} |0\rangle \\ &= q^{-\frac{1}{4}(\nu_1 - \nu_{-1}) - \frac{1}{2}(\nu_1 - \nu_{-1})\nu_1} \frac{(T_1^1)^{\frac{1}{2}(\nu_1 - \nu_{-1})} (T_0^1)^{\nu_{-1}}}{\sqrt{[\nu_1]! [\nu_{-1}]! [2]^{\nu_{-1}}}} |0\rangle. \end{aligned} \quad (55)$$

For the states from the right from the central ladder ( $\nu_1 \leq \nu_{-1}$ ,  $n_1 = 0$ ,  $\max n_0 = \nu_1$ ) we get the expressions

$$\begin{aligned} |0, n_0, n_{-1}\rangle &= q^{-\frac{1}{2}n_{-1} - n_{-1} n_0} \frac{(T_0^1)^{n_0} (T_{-1}^1)^{n_{-1}}}{\sqrt{[2]^{n_0} [n_0]! [2n_{-1} + n_0]!}} |0\rangle \\ &= q^{-\frac{1}{4}(\nu_1 - \nu_{-1}) + \frac{1}{2}\nu_1(\nu_1 - \nu_{-1})} \frac{(T_0^1)^{\nu_1} (T_{-1}^1)^{\frac{1}{2}(\nu_{-1} - \nu_1)}}{\sqrt{[\nu_1]! [\nu_{-1}]! [2]^{\nu_1}}} |0\rangle. \end{aligned}$$

In this case ( $\max n_0 = \nu_1$  or  $\nu_{-1}$ ) the table of the basic states has the form as presented in figure 4.

This case corresponds to a coupling to a maximum degree for the operator  $T_0$ . With it we move along the columns by acting on the minimal-weight state an infinite number of times. These two forms for the basis states are equivalent. The transition between Case 1 and Case 2 is realized by means of the relation

$$(T_0^1)^2 = q^{-1} [2] T_1^1 T_{-1}^1. \quad (56)$$

We now give the action of the  $q$ -deformed tensor representation of the algebra  $su_t(2) \sim so_t(3)$  with generators  $L_m^1$  ( $m = 0, \pm 1$ ). First note that

$$[L_m^1, N] = 0 \quad m = 0, \pm 1.$$

From decomposition (7) one can observe that in each subspace  $\mathcal{H}_\nu$  ( $\nu = 2n$ ) of  $\mathcal{H}_+$  an irreducible representation of  $su_t(2)$  acts. The eigenvalue of the second-order Casimir operator for a given irreducible representation is  $\frac{1}{[2]} [2n][2n+2]$  (41). Furthermore we know the action of the raising and lowering operators  $L_{\pm 1}^1$  for  $n$  fixed in the case when  $n_0 = \max n_0 = \nu_1$  or  $\nu_{-1}$ . Along the rows given by  $\nu = 2n$  at  $\nu_1 \geq \nu_{-1}$  we move by acting with the operator  $(L_{-1}^1)^k$  ( $k \leq n$ ) on the highest-weight state  $|n_1 = n, 0, 0\rangle$

$$(L_{-1}^1)^k |n, 0, 0\rangle = q^{-\frac{1}{2}k(2n-k)} \sqrt{\frac{[2n]! [k]!}{[2n-k]!}} |n-k, k, 0\rangle.$$

Furthermore, for  $\nu_1 \leq \nu_{-1}$

$$(L_{-1}^1)^k |0, n, 0\rangle = q^{\frac{1}{2}k^2} \sqrt{\frac{[n+k]!}{[n-k]!}} |0, n-k, k\rangle$$

and therefore

$$(L_{-1}^1)^{2n}|n, 0, 0\rangle = [2n]!|0, 0, n\rangle.$$

For the action of  $(L_{+1}^1)^k$  ( $k \leq n$ ) on the lowest weight vector  $|0, 0, n\rangle$  at  $\nu_1 \leq \nu_{-1}$  we have that

$$(L_{+1}^1)^k|0, 0, n\rangle = q^{\frac{1}{2}k(2n-k)} \sqrt{\frac{[k]![2n]!}{[2n-k]!}}|0, k, n-k\rangle$$

and from the centre for  $\nu_1 \geq \nu_{-1}$  we obtain the result

$$(L_{+1}^1)^k|0, n, 0\rangle = q^{-\frac{1}{2}k^2} \sqrt{\frac{[n+k]!}{[n-k]!}}|k, n-k, 0\rangle$$

and it therefore follows that

$$(L_{+1}^1)^{2n}|0, 0, n\rangle = [2n]!|n, 0, 0\rangle.$$

The operators  $L_{\pm 1}^1$  do not differ essentially from the operators  $J_{\pm}$  (36) and so their action on the basis states is easily obtained by means of (26) if we take into account the appropriate  $q$ -factors and the relations  $j = \frac{\nu}{2} = n$  and  $j_0 = n_1 - n_{-1}$ . The eigenvalues of the operator  $L_0^1$  (39) on the basis states are given by

$$L_0^1|n_1, n_0, n_{-1}\rangle = \frac{1}{[2]} \{q[2n_1 + n_0][2n_{-1} + n_0 + 1]|n_1, n_0, n_{-1}\rangle - q^{-1}[2n_{-1} + n_0][2n_1 + n_0 + 1]|n_1, n_0, n_{-1}\rangle\}.$$

Unlike  $J_0$ ,  $L_0^1$  has different eigenvalues for each step of a given ladder.

## 6. Conclusions

In this paper the boson representation of  $sp(4, R)$  algebra and two different deformations of it,  $sp_q(4, R)$  and  $sp_t(4, R)$ , were considered. The initial as well as the deformed representations act in the same Fock space  $\mathcal{H}$ . All three are reducible and each one is decomposed into two irreducible representations acting in the subspaces  $\mathcal{H}_+$  and  $\mathcal{H}_-$  of  $\mathcal{H}$ .

The deformed representation  $sp_q(4, R)$  is based on the standard  $q$ -deformation of the two-component Heisenberg algebra, realized in terms of creation and annihilation operators. In this case eight of the ten generators are deformed, but the complete set of the boson number operators  $N_1$  and  $N_{-1}$  (or their linear combinations  $N$  and  $J_0$ ) are preserved as in the 'classical' case. The latter cannot be expressed in terms of deformed bosons. The subalgebras of the boson  $sp(4, R)$  (the compact  $u(2)$  and the noncompact  $u^\varepsilon(1, 1)$  with  $\varepsilon = 0, \pm$ ) are also deformed and their deformed representations are contained in  $sp_q(4, R)$ . They are reducible in the spaces  $\mathcal{H}_+$  and  $\mathcal{H}_-$  and decompose into irreducible ones. In this way a full description of the IURs of  $u_q(2)$  of the deformed ladder series  $u_q^0(1, 1)$  and of two deformed discrete series  $u_q^\pm(1, 1)$  were obtained.

An open question is the possibility of constructing in the  $sp_q(4, R)$  framework other series of representations, for example, by means of non-Fock bosons [20, 21].

The other deformed representation,  $sp_t(4, R)$ , is realized by means of a transformation of the  $q$ -deformed bosons into  $q$ -tensors (spinor-like) with respect to the  $su_q(2)$  operators. Unlike  $sp_q(4, R)$ , the  $sp_t(4, R)$  generators are deformed and have expressions in terms of tensor products of spinor-like operators. The important result in this case is the appearance of a deformed  $su_t(2)$ , which can be interpreted as a deformation of the angular momentum algebra  $so(3)$ . Its representation in  $\mathcal{H}$  is reducible and is decomposed into irreducible ones, giving in this way a full description. In a future application, the dependence of the two quantum number basis states in  $\mathcal{H}_+$  will be presented in terms of three linked integer parameters.



The reductions into subalgebras (compact and noncompact) of  $sp(4, R)$  and its deformations give rise to the possibility of different models with dynamical symmetries. In any physical interpretations of the results it is important to pay attention to the fact that the deformations do not change the basis states in the Fock space, only the action of the operators on them. This, with a view towards applications, gives rise to richer choices for the operators associated with the observables—nondeformed, as well as deformed. In Hamiltonian theory this implies a dependence of the matrix elements on the deformation parameter, leading to the possibility of greater flexibility and richer structures within the framework of algebraic descriptions.

An interesting future development, aimed at physical applications, is the description of the compact and noncompact contents of  $q$ -deformed symplectic algebras of higher dimension [22].

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### References

- [1] Goshen S and Lipkin H J 1959 *Ann. Phys.* **12** 301
- [2] Rosensteel G and Rowe D J 1977 *Phys. Rev. Lett.* **38** 10
- [3] Peterson D R and Hecht K T 1980 *Nucl. Phys. A* **344** 361
- [4] Georgieva A I, Ivanov M I, Raychev P P and Roussev R P 1986 *Int. J. Theor. Phys.* **25** 1181
- [5] Drenska S, Georgieva A, Gueorguiev V, Roussev R and Raychev P 1995 *Phys. Rev. C* **52** 1853
- [6] Castanos O, Chacon E, Moshinsky M and Quesne C 1985 *J. Math. Phys.* **26** 2107
- [7] Klein A and Marshalek E 1991 *Rev. Mod. Phys.* **63** 375
- [8] Asherova R M, Fursa D V, Georgieva A and Smirnov Yu F 1993 *J. Phys. G: Nucl. Phys.* **19** 1887
- [9] Biedenharn L C and Lohe M A 1995 *Quantum Group Symmetry and  $q$ -Tensor Algebras* (Singapore: World Scientific)
- [10] Hayashi T 1990 *Commun. Math. Phys.* **127** 129
- [11] Quesne C and Vansteenkiste N 1998 *Phys. Lett. A* **240** 21  
Quesne C and Vansteenkiste N 1999 *Preprint* ULB/229/CQ/99/1, math-ph/0003025
- [12] Barut A O and Raszhka R 1977 *Theory of Group Representations and Applications* (Warsaw: PWN)
- [13] Dotan Y, Gell-Mann M and Ne'eman Y 1956 *Phys. Lett.* **17** 148
- [14] Kulish P P and Damaskinski E V 1990 *J. Phys. A: Math. Gen.* **23** L415
- [15] Smirnov Yu F, Tolstoi V and Kharitonov Yu 1991 *Sov. J. Nucl. Phys.* **53** 593
- [16] Georgieva A 1993 Tensorial structure of a  $q$ -deformed  $sp(4, R)$  super-algebra *ICTP Trieste preprint* IC/93/50
- [17] Georgieva A and Dankova Ts 1994 *J. Phys. A: Math. Gen.* **27** 1251
- [18] Georgieva A I, Goleminov J D, Ivanov M I and Geyer H B 1999 *J. Phys. A: Math. Gen.* **32** 2403
- [19] Del Sol Mesa A, Loyola A, Moshinsky M and Velazquez V 1993 *J. Phys. A: Math. Gen.* **26** 1147
- [20] Chaichian M, Grosse H and Prešnajder P 1994 *J. Phys. A: Math. Gen.* **27** 2045
- [21] Chaichian M, Gonzelez Felipe R and Prešnajder P 1995 *J. Phys. A: Math. Gen.* **28** 2247
- [22] Green G 1999 *J. Math. Phys.* **40** 1074